Migration-Contagion Processes

F. Baccelli
joint work with S. Foss and S. Shneer

July 4, 2022, Grenoble
AEP 2022

European Research Council
Established by the European Commission

Introduction-1

Role of migration in the propagation of epidemics ?

- simplest possible migration model:
  closed network of $\cdot/\mathbf{M}/\infty$ queues
- simplest epidemic process:
  the SIS model
- in the thermodynamic limit
  to further simplify the problem

Introduction-2

Epidemic interpretation:

- **Individuals move from place (station) to place places are indexed by, say $\mathbb{Z}$**
- **The overall density of individuals (mean number of individuals per station) is η**
- **An individual stays at a place for a random time which is exp. distributed with parameter μ (migration rate)**
- **The departure rate of individuals from a given place is hence $\lambda = \eta\mu$**
- **When it leaves a place, an individual migrates to a place chosen independently and 'at random'**
- **At each station, individuals are subject to the SIS dynamics with parameters α, β**
- **Infections and recoveries take place in stations and conditionally on the stations state (local interaction)**

Introduction-3

Related work:

(I) The Contact Process on Deterministic Graphs

- **In the absence of mobility, the problem was extensively studied in the particle system literature Liggett 85**
- **There is a large corpus of results on infinite graphs with finite degrees such as grids and regular trees**
- **On finite graphs, the main question is that of the phase transition between a logarithmic and an exponential growth of the time till extinction**
- **This was studied on deterministic graphs like finite grids and regular trees**

Introduction-4

(II) SIS on Finite Random Graphs

- **Overview:**
  R. Pastor-Satorras, C. Castellano, P. Van Mieghem, and A. Vespignan (Rev. Mod. Phys., 2015)
- **Book: M.E.J. Newman, 2015**
- **Review on the moment closure techniques:**
  C. Kuehn (Control of Self-Organizing Nonlinear Systems, 2016)

Introduction-5

(III) The Contact Process on Infinite Random Graphs

The contact process was also studied on infinite random graphs with unbounded degrees

- **supercritical Bienaymé-Galton-Watson tree**
  R. Pemantle (Ann. Probab., 1992) where it was shown that some critical values can be degenerate
- **Euclidean point processes**
  - **G. Ganesan (Adv. Appl. Probab., 2015)**
  - **C.V. Hao (Combin., Probab. and Computing, 2018)**
  - **L. Menard and A. Singh (Ann Sci ENS, 2016)**

Introduction-6

(IV) Mobile SIS Epidemics

- Basic model for SIS on graphs: agents perform a random walk on the random graph and agents meeting at a given point of the graph may infect each other D. Figueiredo, G. Iacobelli, and S. Shneer (J. Stat.Physics, 2020) and the references therein
- On point processes
  a computational framework for evaluating the role of mobility on the propagation of epidemics, F. Baccelli and N. Ramesan, (J. Math. Biology, January 2022)

Outline

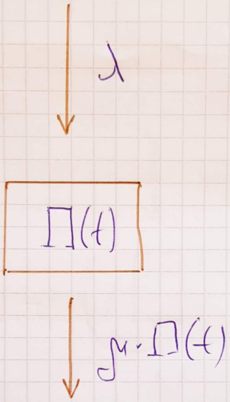
- **(0) Preliminaries: $\mathbf{M}/\mathbf{M}/\infty$ queue and networks**
- **(1) SIS (Susceptible-Infected-Susceptible) Reactor**
- **(2) Closed Network of SIS Reactors:**
  - **Its TL (Thermodynamic Limit)**
  - **TL Phase Diagram w.r.t. Population Density**
- **(3) Related Models (DOCS and AIR, and their TL's)**
- **(4) Comparison of Models**
- **(5) Dependence on Migration Rate**
- **(6) Open Questions**

(0.1) M/M/∞ Queue

- **Input rate λ, service rate μ, number of customers $\Pi_z(t)$, $t \geq 0$, where $z = \Pi_z(0)$**
- **Positive recurrent continuous time Markov process with countably many states**
- **Unique stationary distribution (stationary process) $\Pi(t), t \geq 0$ with $\Pi(t) \sim \text{Poisson}(\lambda/\mu)$**
- **For all initial conditions z, exponentially fast coupling convergence:**

  $$P(\Pi_z(u) = \Pi(u) \text{ for all } u \geq t) \geq 1 - C_1 \exp(-C_2 t), \quad t \geq 0$$

  where C_1, C_2 depend on z



$$\Pi(t) \sim \mathrm{Pois}(\eta)$$
$$\eta = \lambda/\mu$$

Fig. 1 M/M/∞ queue

(0.2) Closed symmetric network of $M/M/\infty$ queues

- N queues, $K \equiv K(N)$ customers, service rate μ
- Next station chosen at random (w.p. $1/N$)
- No exogenous input / no exogenous output
- Finite symmetric Markov chain, unique stationary distribution which is multinomial (Gordon-Newell),

$$P(\bigcap_{i=1}^{N}\{\Pi_i(t) = k_i\}) = \frac{K!}{\prod_{i=1}^{N} k_i!} \cdot \frac{1}{N^K}$$

  where $\sum k_i = K$
- Exponential convergence rate, for any initial state

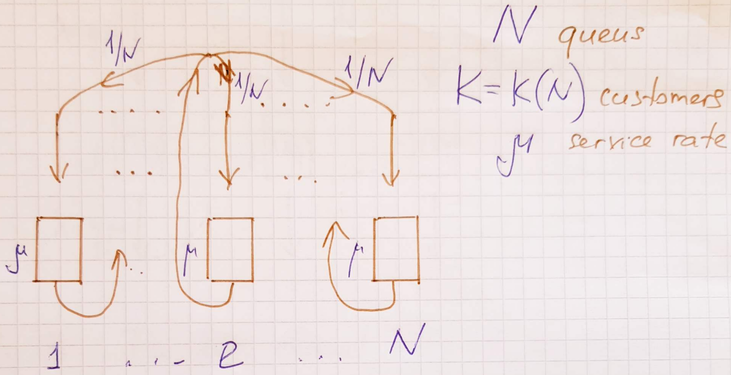


Fig. 2 Closed network of $M/M/\infty$ queues

(0.3) Thermodynamic Limit (TL) for the closed network of M/M/∞ queues

Let $\mathbf{N} \to \infty$, $\mathbf{K}/\mathbf{N} \to \eta \in (\mathbf{0}, \infty)$

- **For any fixed $\mathbf{r} = \mathbf{1}, \mathbf{2}, \ldots$, the random processes $\mathbf{\Pi_1(t)}, \ldots, \mathbf{\Pi_r(t)}$, $\mathbf{t} \geq \mathbf{0}$ are asymptotically independent (as processes) and any of them is as in the single M/M/∞ queue with input rate $\lambda := \eta\mu$ and service rate μ**
- **The natural parameterization is (η, μ), then $\lambda = \eta\mu$**

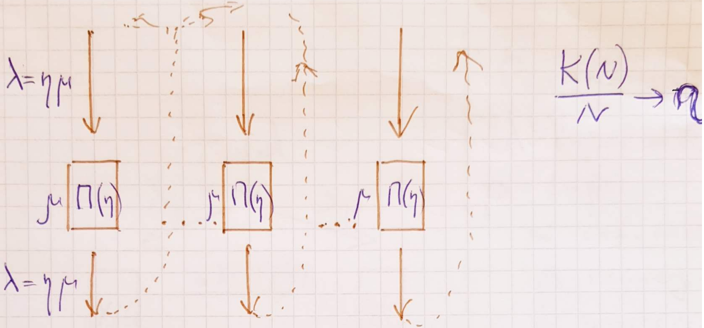


Fig. 3 TL for closed network of M/M/∞ queues.

(1.1) SIS Reactor

$M/M/\infty$ queue, input rate λ, service rate μ

Now:

- **each customer is either *susceptible* (S) or *infected* (I)**
- **$X(t)$, number of susceptible**
- **$Y(t)$, number of infected**
- $X(0) \equiv X_x(0) = x, Y(0) \equiv Y_y(0) = y, X(t) + Y(t) = \Pi(t)$

Input:

customer *infected* w.p. p, *susceptible* w.p. $q = 1 - p$

Infection-recovery mechanism:

- **any customer:**
  S $\to$ I with rate $\alpha Y(t)$, and I $\to$ S with rate β
- **total:**
  S $\to$ I with rate $\alpha X(t)Y(t)$, and I $\to$ S with rate $\beta Y(t)$

Comment:

If we do not distinguish I and S, this is an $M/M/\infty$ queue

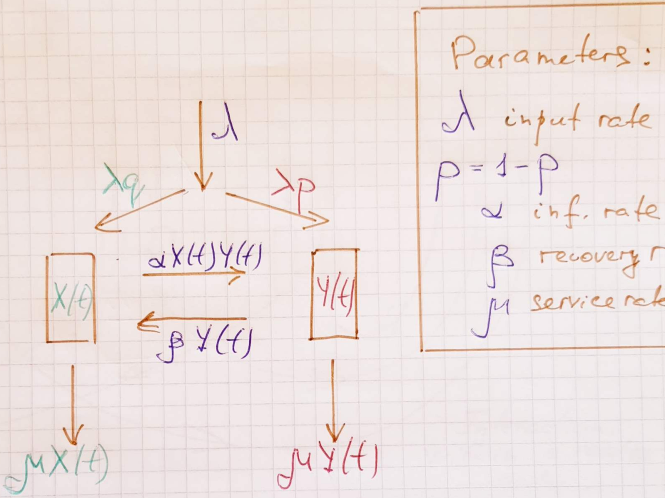


Fig. 4 SIS reactor

(1.2) SIS Reactor

Vector $(\mathbf{X}(t), \mathbf{Y}(t))$ is again positive recurrent, unique stationary distribution, exponential convergence rate

- No product form
- Neither $\mathbf{X}(t)$ nor $\mathbf{Y}(t)$ Poisson
- Their means are unknown

Wave type PDE for Generating Function
$\mathbf{\Phi}(\mathbf{x}, \mathbf{y}) = \mathbf{E}[\mathbf{x}^{\mathbf{X}}\mathbf{y}^{\mathbf{Y}}]$ of stationary process:
for all $\mathbf{0} \le \mathbf{x} \le \mathbf{1}, \mathbf{0} \le \mathbf{y} \le \mathbf{1}$,

$$
\begin{aligned}
&(\lambda \mathbf{q}(\mathbf{1}-\mathbf{x}) + \lambda \mathbf{p}(\mathbf{1}-\mathbf{y}))\mathbf{\Phi}(\mathbf{x}, \mathbf{y}) \\
=&\mu(\mathbf{1}-\mathbf{x})\mathbf{\Phi}_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) + (\mu(\mathbf{1}-\mathbf{y}) + \beta(\mathbf{x}-\mathbf{y}))\mathbf{\Phi}_{\mathbf{y}}(\mathbf{x}, \mathbf{y}) \\
+&\alpha \mathbf{y}(\mathbf{y}-\mathbf{x})\mathbf{\Phi}_{\mathbf{xy}}(\mathbf{x}, \mathbf{y})
\end{aligned}
$$

We could not solve this PDE

(1.3) SIS Reactor

Rate Conservation Identities:

First order:

$$\lambda \mathbf{p} + \alpha \mathbf{E}[\mathbf{XY}] = (\mu + \beta)\mathbf{E}[\mathbf{Y}] \qquad \text{or, equivalently,}$$

$$\lambda \mathbf{q} + \beta \mathbf{E}[\mathbf{Y}] = \mu \mathbf{E}[\mathbf{X}] + \alpha \mathbf{E}[\mathbf{XY}]$$

Second order:

$$(\lambda \mathbf{p} + \mu + \beta)\mathbf{E}[\mathbf{Y}] + \alpha \mathbf{E}[\mathbf{XY^2}] = (\mu + \beta)\mathbf{E}[\mathbf{Y^2}],$$

$$(\lambda \mathbf{q} + \mu)\mathbf{E}[\mathbf{X}] + (\alpha + \beta)\mathbf{E}[\mathbf{XY}] = \alpha \mathbf{E}[\mathbf{X^2Y}] + \mu \mathbf{E}[\mathbf{X^2}]$$

Higher order: ...

Conjectures: anti-association (numerical evidence)...

(1.4) SIS Reactor

Fraction of infected: $\mathbf{p}$ **in input;** $\mathbf{p_o} = \frac{\mu \mathbf{EY}}{\lambda}$ **in output**

(1) Fix strictly positive λ, μ, α**, and** β

Input-output map $\mathbf{p} \to \mathbf{p_o} \equiv \mathbf{g(p)}$

Thanks to coupling arguments:

(i) **Function** $\mathbf{g(p)}$ **is strictly monotone increasing, strictly concave, and differentiable,** $\mathbf{g(0) = 0}$ **and** $\mathbf{g(1) < 1}$

(ii) **Fixed-point equation** $\mathbf{p = g(p)}$

- **has only one solution** $\mathbf{p = 0}$ **iff** $\mathbf{g'(0) \leq 1}$
- **has two solutions** $\mathbf{p = 0}$ **and** $\mathbf{p^* > 0}$ **iff** $\mathbf{g'(0) > 1}$

(1.5) SIS Reactor

(2) Fix now any strictly positive μ, α, β and consider $\mathbf{g}'(\mathbf{0})$ as a function of $\eta = \lambda/\mu$

(iii) The value of $\mathbf{g}'(\mathbf{0})$ is a strictly increasing function of η

(iv) There is a finite and strictly positive $\eta^{(s)} = \eta^{(s)}(\mu, \alpha, \beta)$ such that $\mathbf{g}'(\mathbf{0}) = \mathbf{1}$

In particular,

$$\frac{\beta}{2\mu + 5\beta} \leq \eta_c^{(s)} \leq \frac{(2(\mu+\beta)(\alpha+2\mu+\beta) - \alpha\beta)\,\beta}{2\alpha\mu(\mu+\beta)}$$

- **The lower bound does not depend on α**
- **These bounds may easily be improved**

(2.1) Closed Network of SIS Reactors

Network:

- **N queues, $K \equiv K(N)$ customers, service rate μ**
- **next station at random (w.p. $1/N$), no input/no output**

On top of that:

- **at time 0:**
  K_I infected and K_S susceptible, $K = K_I + K_S$
- **SIS mechanism in each queue**

Finite symmetric Markov chain, unique stationary distribution, exponential convergence rate

Eventually: all customers become susceptible

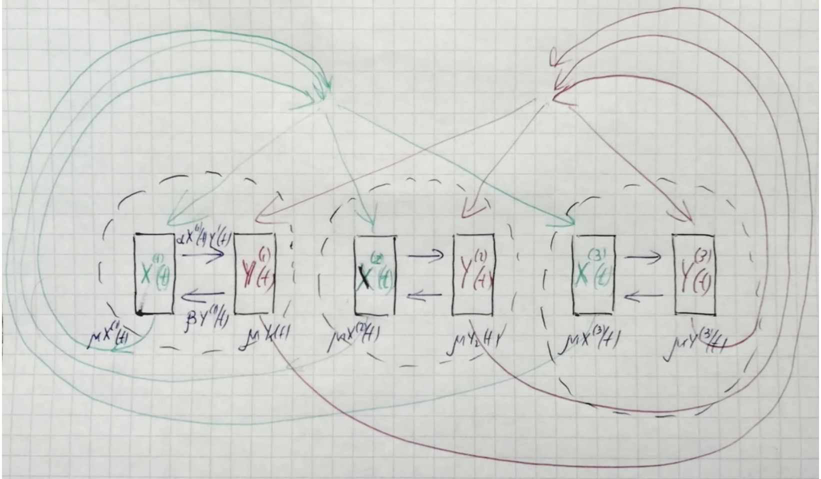
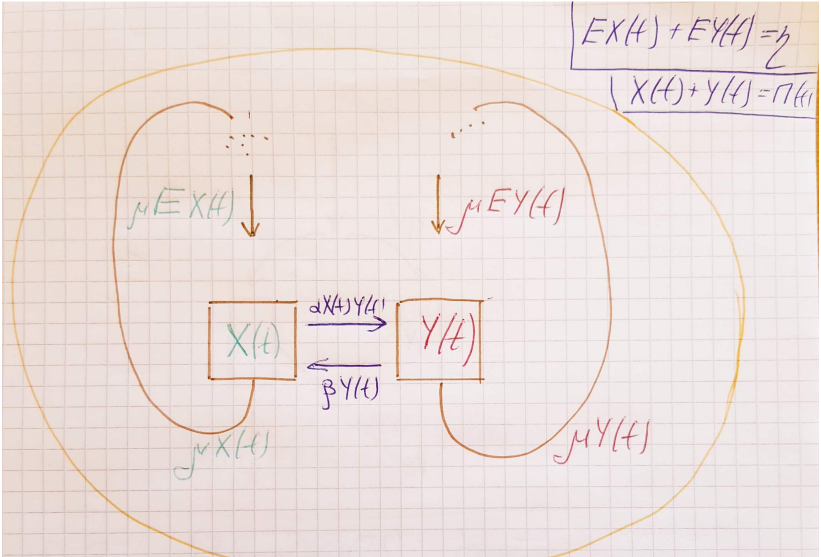


Fig. 5. Closed network of $N=3$ SIS reactors.

(2.2) Thermodynamic Limit (TL) of Closed Networks of SIS Reactors

Let $\mathbf{N} \to \infty$, $\mathbf{K}/\mathbf{N} \to \eta \in (\mathbf{0}, \infty)$



(2.3) On TL of Closed Networks

Thermodynamic propagation of chaos ansatz

- **The TLs pertain to a family of closed networks as above**
- **They are all infinite-dimensional Markov systems**
- **Can be seen as certain non-hom. Markov processes**
- **Broad range of asymptotic behaviours possible**

We assume that the thermodynamic limit exists and satisfies the following properties on any compact of time:

- **Stations have independent dynamics**
- **In each station, the arrival point process of S (resp. I) is (possibly non-homogeneous) Poisson, with these two processes being independent**

This set of properties will be referred to as the Thermodynamic propagation of chaos ansatz

(2.4) On TL of Closed Networks

Definition

In any TL, we will say that

- **there is survival if the associated Markov system has a steady state distribution with a fraction $0 < p < 1$ of susceptible customers**
- **there is weak extinction if there is no such p**
- **there is strong extinction if, for all initial conditions, the associated Markov system converges to a regime with no infected customers**

(2.5) TL of Closed Networks of SIS Reactors

Back to SIS Reactors:

Let $\eta^{(\mathbf{s})} = \eta^{(\mathbf{s})}(\mu, \alpha, \beta)$ **be the solution to** $\mathbf{g}'(\mathbf{0}) = \mathbf{1}$

Theorem
If the SIS thermodynamic propagation of chaos ansatz holds, then, in the SIS thermodynamic limit,

- **there is survival if** $\eta > \eta_c^{(s)}$
- **there is strong extinction if** $\eta \leq \eta_c^{(s)}$

(3.1) SIS-DOCS Reactor

Here DOCS = Departure on Change of State

- **The SIS-DOCS reactor features a single station like in the SIS case**
- **The departure rate of susceptible customers and of infected customers is μ**
- **The infection mechanism of the SIS model is replaced by a simultaneous infection and departure mechanism with the following characteristics:**
  - **if the number of infected is $\mathbf{Y(t)}$, each susceptible gets infected with rate $\alpha\mathbf{Y(t)}$ and, upon infection, it immediately leaves the system for good**
  - **each infected recovers with rate β and, upon infection, it immediately leaves the system for good**

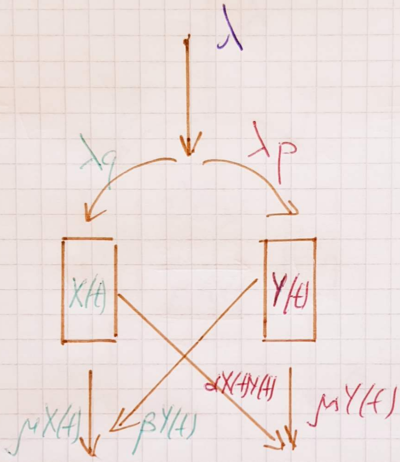


Fig. 7 SIS-DOCS reactor

(3.2) SIS-DOCS Reactor

The associated PDE for the stationary generating function of the general SIS-DOCS reactor is

$$\begin{aligned}(\lambda \mathbf{q}(\mathbf{1}-\mathbf{x})+\lambda \mathbf{p}(\mathbf{1}-\mathbf{y}))\mathbf{\Phi}(\mathbf{x},\mathbf{y}) \\ = \mu(\mathbf{1}-\mathbf{x})\mathbf{\Phi}_{\mathbf{x}}(\mathbf{x},\mathbf{y}) + \nu(\mathbf{1}-\mathbf{y})\mathbf{\Phi}_{\mathbf{y}}(\mathbf{x},\mathbf{y}) \\ +\alpha \mathbf{y}(\mathbf{1}-\mathbf{x})\mathbf{\Phi}_{\mathbf{xy}}(\mathbf{x},\mathbf{y})\end{aligned}$$

Here $\nu = \mu + \beta$

This wave-type equation can be solved

(3.3.) Properties of SIS-DOCS Reactor

- **The process of infected customers is autonomous**
- **The stationary distribution of Y is Poisson** λ/ν
- **The distribution of X is not Poisson**
- **The solution of the PDE gives**

$$\mathbf{EX} = \frac{\lambda \mathbf{q}}{\nu + \alpha} \int_0^1 \mathbf{e}^{-\frac{\lambda \mathbf{p} \alpha^2}{\nu(\nu+\alpha)^2}(1-\mathbf{t})} \mathbf{t}^{\frac{\mu}{\nu+\alpha} + \lambda \mathbf{p} \frac{\alpha}{(\nu+\alpha)^2} - 1} \mathbf{dt}$$

(3.4) Properties of SIS-DOCS Reactor

- **Threshold $\eta_{\mathbf{c}}^{(\mathbf{d})}$ such that the fixed-point equation $\mathbf{p_o} \equiv \mathbf{g(p)} = \mathbf{p}$ has a non-zero solution iff $\eta > \eta_{\mathbf{c}}^{(\mathbf{d})}$:**

$$\eta_{\mathbf{c}}^{(\mathbf{d})} = \frac{\beta}{\alpha}\left(\mathbf{1} + \frac{\alpha}{\mathbf{2}\mu + \beta}\right)$$

- **$\mathbf{X}$ and $\mathbf{Y}$ are dependent**
- **Their sum is stochastically smaller than a Poisson random variable with parameter λ/μ**
- **Simple conservation equations:**

$$\lambda \mathbf{p} = \nu \mathbf{EY}$$
$$\lambda \mathbf{q} = \mu \mathbf{EX} + \alpha \mathbf{EXY}$$

(3.5) Closed Network of SIS-DOCS Reactors

Closed queuing network with N stations and K customers

- **If station n has $X^{(n)}(t)$ susceptible and $Y^{(n)}(t)$ infected customers, each susceptible customer swaps to infected with the instantaneous rate $\alpha Y^{(n)}(t)$ and upon infection, it simultaneously leaves this station and is routed to one of the N stations chosen at random**
- **Each infected customer becomes susceptible with rate β; upon recovery, it simultaneously leaves and is routed to a station chosen at random**
- **As in the closed SIS network model, each customer (infected or susceptible) also leaves the station with a departure rate μ and is then also routed to a station chosen at random**

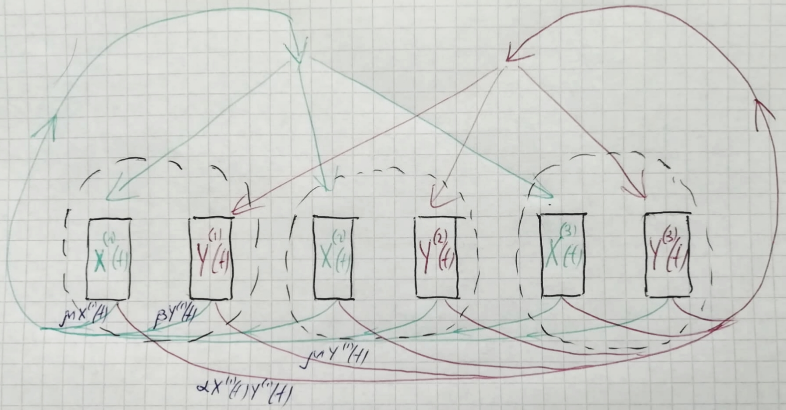
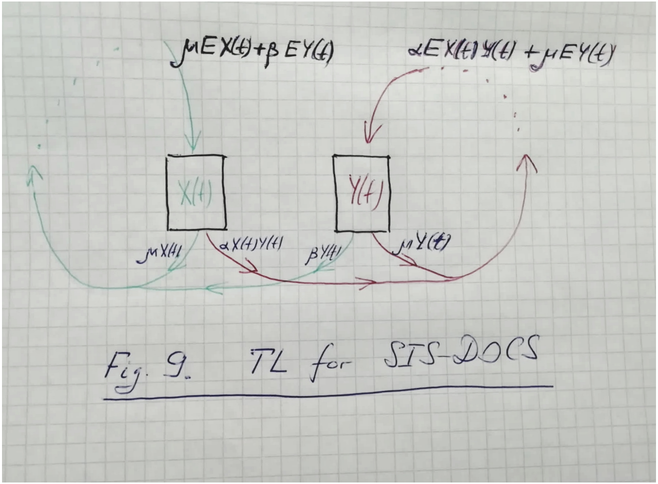


Fig. 8. Closed network of $N=3$ SIS-DOCS reactors

(3.6) TL of closed SIS-DOCS networks

We again let N tend to infinity and K/N $\to \eta$



(3.7) TL of closed SIS-DOCS networks

In the thermodynamic limit, we have with necessity

$$\alpha \mathbf{EXY} = \beta \mathbf{EY}$$

This obviously holds for $\mathbf{Y} = \mathbf{0}$ a.s.
Based on the analysis of the exact expression for $\mathbf{EX}$ in the SIS-DOCS reactor, we get:

Theorem
In the SIS-DOCS thermodynamic limit, if $\beta < \mu$, then

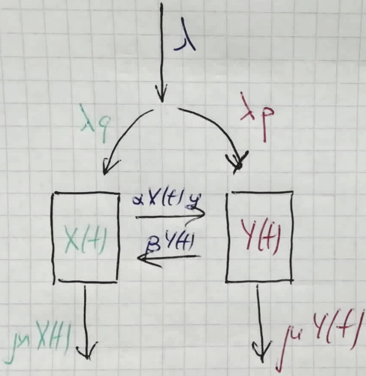
- **there is survival if $\eta > \eta_{\mathbf{c}}^{(\mathbf{d})}$**
- **there is weak extinction if $\eta \leq \eta_{\mathbf{c}}^{(\mathbf{d})}$**

Comment: Condition $\beta < \mu$ is just technical, and we conjecture that the same result holds without it

(3.8) SIS-AIR Reactor

AIR = Averaged Infection Rate.

- **Open network of two M/M/∞ stations**
- **Input rate λ**
- **Each customer is infected with probability $\mathbf{p}$ or susceptible with probability $\mathbf{q = 1 - p}$**
- **Service rate of any customer is μ**
- **Every infected become susceptible with rate β**
- **Difference with SIS: every susceptible becomes infected with rate $\alpha\mathbf{y}$, where y is another parameter**



y is a parameter

Fig.10. SIS - AIR reactor

(3.9) SIS-AIR Reactor

- **Open Jackson network with product-form stationary distribution, which is the product of two Poisson distributions, with parameters $\frac{\lambda_1}{\mu}$ and $\frac{\lambda_2}{\mu}$, correspondingly**
- **From traffic equations, we get**

$$\lambda_1 = \lambda \mathbf{q} + \lambda_2 \frac{\beta}{\mu + \beta}$$
$$\lambda_2 = \lambda \mathbf{p} + \lambda_1 \frac{\alpha \mathbf{y}}{\mu + \alpha \mathbf{y}}$$

and

$$\lambda_1 = \frac{(\mu + \alpha y)(\beta + \mu q)\lambda}{(\mu + \beta)(\mu + \alpha y) - \beta \alpha y}$$

- **Further,** $\mathbf{EX} = \frac{\lambda_1}{\mu + \alpha \mathbf{y}} = \frac{\lambda}{\mu} - \mathbf{y}$

(3.10) Closed SIS-AIR Network

Consider a closed network with $\mathbf{N}$ stations and $\mathbf{K}$ customers
Let

$$\widehat{\mathbf{Y}}(\mathbf{t}) = \frac{1}{\mathbf{N}} \sum_{\mathbf{i}=1}^{\mathbf{N}} \mathbf{Y}^{(\mathbf{i})}(\mathbf{t})$$

$\mathbf{Y}^{(\mathbf{i})}(\mathbf{t})$**: number of infected customers at station $\mathbf{i}$ at time $\mathbf{t}$**

Difference with SIS network: averaged infection rates: at station $\mathbf{i}$: $\alpha \mathbf{X}_\mathbf{i}(\mathbf{t})\widehat{\mathbf{Y}}(\mathbf{t})$ in place of $\alpha \mathbf{X}_\mathbf{i}(\mathbf{t})\mathbf{Y}_\mathbf{i}(\mathbf{t})$

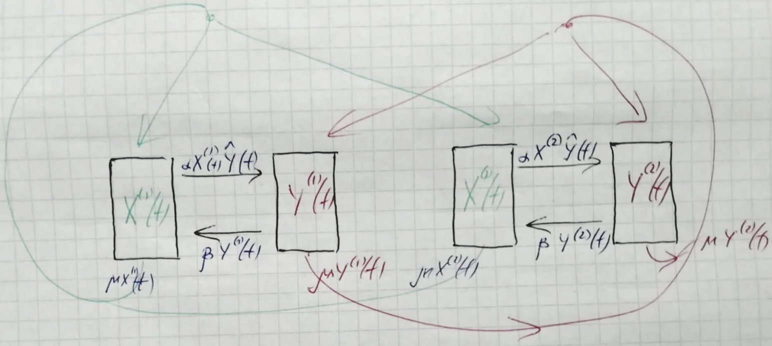


Fig. 11. Closed network of N=2 SIS-AIR reactors.

(3.11) TL of Closed SIS-AIR Networks

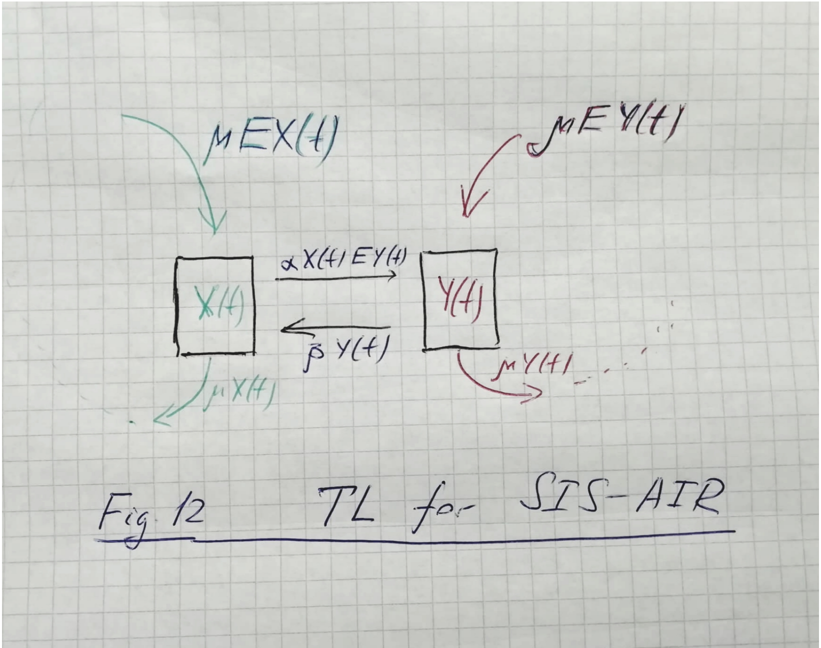


Fig 12 TL for SIS-AIR

(3.12) TL for SIS-AIR

Let $\eta_{\mathbf{c}}^{(\mathbf{a})} := \frac{\beta}{\alpha}$

Theorem

Under SIS-AIR thermodynamic propagation of chaos ansatz

- **if** $\eta \leq \eta_{\mathbf{c}}^{(\mathbf{a})}$**, there is weak extinction**
- **if** $\eta > \eta_{\mathbf{c}}^{(\mathbf{a})}$**, there is survival**

Given survival, in the stationary regime of the TL

$$\mathbf{EX} = \frac{\beta}{\alpha}$$

$$\mathbf{EY} = \eta - \frac{\beta}{\alpha}$$

$$\mathbf{1} - \mathbf{p}^* = \mathbf{q}^* = \frac{\beta}{\eta\alpha}$$

X and Y are independent and Poisson

(4.1) System Comparison

We compare the TLs of the 3 models

- **analytically**
- **numerically**

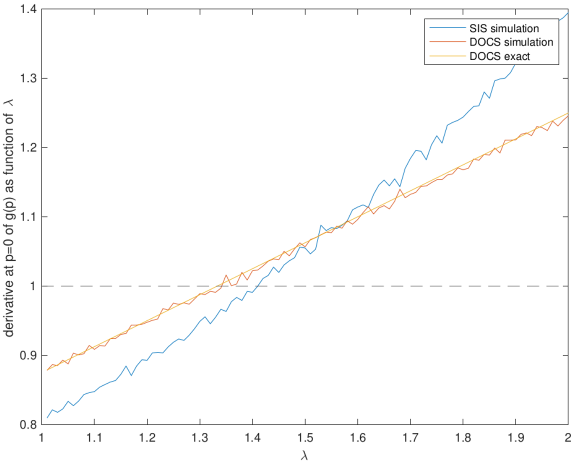
We will say that system A is safer than system B if

$$\eta_{\mathbf{c}}^{(\mathbf{A})} \geq \eta_{\mathbf{c}}^{(\mathbf{B})}$$

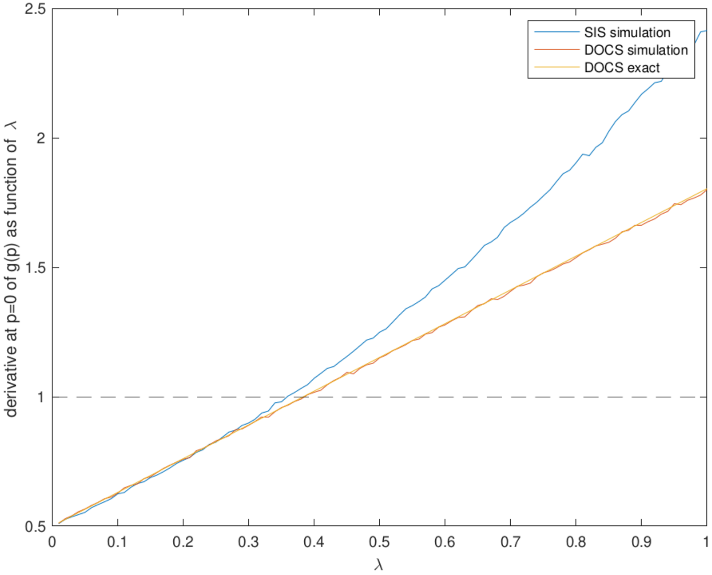
(4.2) Comparison of SIS TL and DOCS TL

Depending on parameters, either SIS or SIS-DOCS is safer

Comparison of $\eta_c^{(s)}$ and $\eta_c^{(d)}$ for $\mu = 1$, $\beta = 1$, and $\alpha = 1$



Comparison of $\eta_c^{(s)}$ and $\eta_c^{(d)}$ for $\mu = 1$, $\beta = 1$, and $\alpha = 20$



(4.3) Comparison of SIS TL and AIR TL

Plain SIS is safer than SIS-AIR

$$\eta_{\mathbf{c}}^{(\mathbf{s})} \geq \eta_{\mathbf{c}}^{(\mathbf{a})} \equiv \frac{\beta}{\alpha}$$

if there is a negative correlation between X and Y
The latter is our conjecture, supported by simulations
This is interesting because AIR is the type of mean-fields that epidemiologists use in their modelling

(4.4) Comparison of SIS-DOCS TL and AIR TL

It directly follows from the expressions of $\eta_{\mathbf{c}}$s that $\eta_{\mathbf{c}}^{(\mathbf{d})} > \eta_{\mathbf{c}}^{(\mathbf{a})}$

This means that SIS-DOCS is safer than AIR

(5) Dependence on Migration Rate

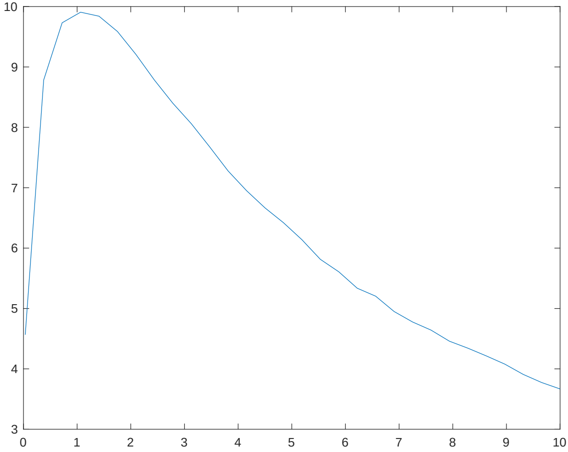
Fix strictly positive η, α, and β
Study the dependence in μ
Partial results for

- **Reactors**
- **TLs**

(5.1) **SIS Reactor**

No monotonicity of $g'(0)$ in μ in general

here $\eta = 3, \beta = 1, \alpha = 5$



(5.2) Thermodynamic Limits

- **SIS: open question**
- **DOCS:**

$$\eta_{\mathbf{c}}^{(\mathbf{d})} = \frac{\beta}{\alpha}\left(\mathbf{1} + \frac{\alpha}{\mathbf{2}\mu + \beta}\right)$$

is a decreasing function of μ:
increasing motion makes system less safe

- **AIR:**

$$\eta_{\mathbf{c}}^{(\mathbf{a})} = \frac{\beta}{\alpha}$$

does not depend on μ

(6) Ongoing Research

- **Proof of the Poisson hypothesis (ansatz)**
- **Full SIS phase diagram for other parameters than population density**
- **SIRS model (S $\rightarrow$ I $\rightarrow$ R $\rightarrow$ S)**
- **Other spatial reactors with mobile customers**