The Joint Spectral Radius Approach to the Convergence of Best Response Dynamics in Routing Games

Olivier Brun, Balakrishna J. Prabhu,
Tatiana Seregina, Morgan Patty

LAAS–CNRS, Toulouse, France

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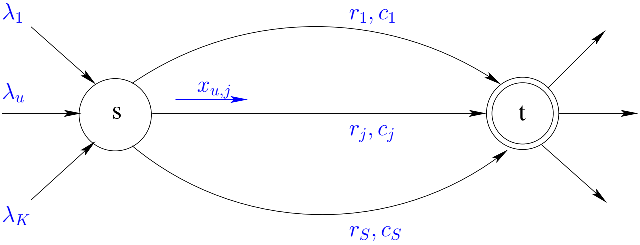
Outline

Routing Game over Parallel Links and Best Response Dynamics

Atomic Routing Game

- Routing game between K users over S parallel links



- Routing strategy of user u: $\mathcal{X}_u = \left\{ \mathbf{x}_u \in \mathbb{R}_+^S \ : \ \sum_{j \in \mathcal{S}} x_{u,j} = \lambda_u \right\}$.
- Strategy profile: $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_K) \in \mathcal{X}$, where $\mathcal{X} = \otimes_u \mathcal{X}_u$
- The cost per unit of traffic of link j is $\frac{c_j}{r_j} \phi(\rho_j)$

Best Response of a Player

- Each user u controls how its own traffic is splitted over the parallel links and seeks to minimize its own cost

$$\text{minimize } T_u(\mathbf{x}_u^*, \mathbf{x}_{-u}) = \sum_{j \in \mathcal{S}} \frac{c_j}{r_j} x_{u,j}^* \, \phi(\rho_j) \qquad \text{(BR-}u\text{)}$$

subject to

$$\mathbf{x}_u^* \in \mathcal{X}_u \text{ and } \rho_j < 1 \text{ for all } j \in \mathcal{S}.$$

- Solution known as the best response of user u at $\mathbf{x}$
- The marginal costs $\frac{\partial T_u}{\partial x_{u,j}}(\mathbf{x}_u^*, \mathbf{x}_{-u})$ are minimal for all links j used by player u in its best response.

Nash Equilibrium

- $x^{(u)}(\mathbf{x})$ is the point reached from $\mathbf{x}$ after the best response of user u

$$x^{(u)}(\mathbf{x}) = (\mathbf{x}_u^*, \mathbf{x}_{-u})$$

- A point $\mathbf{x}^*$ is Nash Equilibrium if

$$\mathbf{x}^* = x^{(u)}(\mathbf{x}^*)$$

  for all u.

- Under some assumptions on $\phi()$, existence and uniqueness of Nash Equilibrium follow from [ORS93]

☞ A. Orda et al., Competitive routing in multi-user communication networks, IEEE/ACM ToN, Oct. 1993.

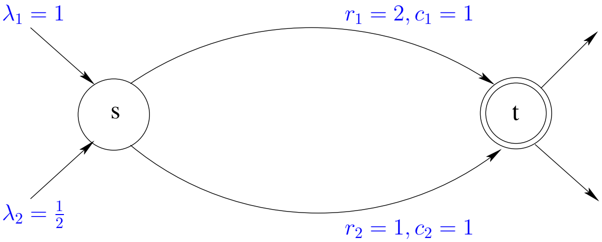
(Sequential) Best Response Dynamics

- Players update their strategies sequentially in the order $1, 2, \ldots, K$
- Let $\hat{x}^{(1)}(\mathbf{x}_0)$ be the strategy profile reached from $\mathbf{x}_0$ after one round

$$\hat{x}^{(1)}(\mathbf{x}_0) = x^{(K)} \circ x^{(K-1)} \circ \ldots \circ x^{(1)}(\mathbf{x}_0)$$

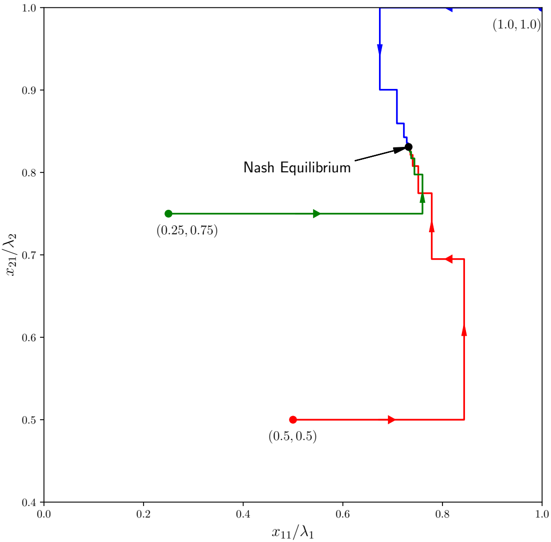
- Similarly, $\hat{x}^{(n)}(\mathbf{x}_0)$ is the strategy profile reached after n rounds
- **Question:** does $\hat{x}^{(n)}(\mathbf{x}_0)$ converge as $n \to \infty$?

Example



- We assume $\phi(x) = \frac{1}{1-x}$
- Explicit expression for the best response of user u

Example



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.0
0.2
0.4
0.6
0.8
1.0
x_{21}/λ_2
x_{11}/λ_1
(1.0, 1.0)
Nash Equilibrium
(0.25, 0.75)
(0.5, 0.5)

Existing Convergence Results

- Two parallel links and two players [ORS93]
- Two parallel links and an arbitrary number of players assuming a linear latency function [ABJS01]
- Two players and an arbitrary number of parallel links [Mer09]

☞ A. Orda et al., Competitive routing in multi-user communication networks, IEEE/ACM ToN, Oct. 1993.

☞ E. Altman et al., Routing into two parallel links: Game-theoretic distributed algorithms, Journal of Parallel and Distributed Computing, 2001.

☞ G.B. Mertzios, Fast convergence of routing games with splittable flows, 2nd Int. Conf. on Theoretical and Mathematical Foundations of Computer Science, 2009.

The Joint Spectral Radius Approach

The Potential Function Approach

- A game is an (exact) potential game [MS96] if there exists $F : \mathcal{X} \to \mathbb{R}$ such that for all players u

$$T_u(\mathbf{x}'_u, \mathbf{x}_{-u}) - T_u(\mathbf{x}_u, \mathbf{x}_{-u}) = F(\mathbf{x}'_u, \mathbf{x}_{-u}) - F(\mathbf{x}_u, \mathbf{x}_{-u})$$

- Convergence of BR dynamics to a Nash Equilibrium [Dur18]
- The symmetric routing game is a potential game [BP16]

$$F(\mathbf{x}) = \sum_{j \in \mathcal{S}} c_j \rho_j \phi(\rho_j) + (K-1) \int_0^{\rho_j} c_j \, \phi(z) dz$$

- Potential not easy to find for asymmetric games

☞ D. Monderer and L.S. Shapley, Potential Games, Games and Economic Behavior, 14, 1996.

☞ S. Durand, Analysis of Best Response Dynamics in Potential Games, PhD thesis, Grenoble, 2018.

☞ O. Brun and B.J. Prabhu, Worst-case analysis of non-cooperative load balancing, Ann. Oper. Res., 239(2), 2016.

The Joint Spectral Radius Approach

- For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, the Mean-Value Theorem implies that

$$
\begin{aligned}
\|\hat{x}^{(n)}(\mathbf{y}) - \hat{x}^{(n)}(\mathbf{x})\| &= \left\| \int_0^1 D\hat{x}^{(n)}(\mathbf{x} + t\mathbf{h}) \cdot \mathbf{h} \, dt \right\|, \text{ where } \mathbf{h} = \mathbf{y} - \mathbf{x}, \\
&\le \int_0^1 \left\| D\hat{x}^{(n)}(\mathbf{x} + t\mathbf{h}) \right\| \, \|h\| \, dt, \\
&\le \varnothing(\mathcal{X}) \int_0^1 \left\| \prod_{i=1}^{n} D\hat{x}^{(1)}(\mathbf{z}_i(t)) \right\| dt, \\
&\le \varnothing(\mathcal{X}) \sup_{M_i \in \mathcal{J}} \|M_n \ldots M_1\|,
\end{aligned}
$$

where $\mathcal{J}$ is the set of one-round Jacobian matrices

- Asymptotic convergence provided that products of n Jacobian matrices converge in norm to 0 as $n \to \infty$ [MPXY07]

☞ Mak et al., A new stability criterion for discrete-time neural networks: Nonlinear spectral radius, Chaos, Solitons and Fractals, 31, 2007.

Joint Spectral Radius

- For a matrix M, the growth rate of $\|M^n\|$ is characterized by the spectral radius $\rho(M)$: $\rho(M)^n = \rho(M^n) \approx \|M^n\|$
- The Joint Spectral Radius [RS60] of $\mathcal{J}$

$$\rho(\mathcal{J}) = \limsup_{n\to\infty} \max_{M_i \in \mathcal{J}} \|M_n \dots M_1\|^{\frac{1}{n}}$$

coincides [BW92] with its Generalized Spectral Radius [DL92]

$$\bar{\rho}(\mathcal{J}) = \limsup_{n\to\infty} \max_{M_i \in \mathcal{J}} \rho(M_n \dots M_1)^{\frac{1}{n}}$$

- Show that the JSR of the set $\mathcal{J}$ of one-round Jacobian matrices is strictly smaller than 1

☞ Rota & Strang, A note on the joint spectral radius, Indag. Math 22, 1960.

☞ Berger & Wang, Bounded semigroups of matricess, Linear Algebra Appl., 166, 1992.

☞ Daubechies & Lagarias, Sets of matrices all infinite products of which converge, Lin. Alg. and its App., 161, 1992.

Structure of Jacobian Matrices

Structure of the Jacobian matrices

- $x^{(u)}$ is differentiable at every point $\mathbf{x} \in \mathcal{X}$ such that no link is marginally used by player u in its best-response at point $\mathbf{x}$

- The Jacobian of $x^{(u)}$ at $\mathbf{x} \in \mathcal{X}$ is defined as

$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_u}(\mathbf{x}) & \dots & \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \\ \vdots & \ddots & & & \vdots \\ \frac{\partial x_u^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_u^{(u)}}{\partial \mathbf{x}_u}(\mathbf{x}) & \dots & \frac{\partial x_u^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \\ \vdots & & & \ddots & \vdots \\ \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_u}(\mathbf{x}) & \dots & \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \end{pmatrix},$$

where the (v, w)-block is

$$\frac{\partial x_v^{(u)}}{\partial \mathbf{x}_w}(\mathbf{x}) = \left(\frac{\partial x_{v,i}^{(u)}}{\partial x_{w,j}}(\mathbf{x}) \right)_{i \in \mathcal{S}, j \in \mathcal{S}},$$

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$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} I & \dots & 0 & \dots & 0 \\ \vdots & \ddots & & & \vdots \\ \frac{\partial x_u^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & 0 & \dots & \frac{\partial x_u^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & \dots & I \end{pmatrix},$$

where the (v, w)-block is

$$\frac{\partial x_v^{(u)}}{\partial \mathbf{x}_w}(\mathbf{x}) = \left(\frac{\partial x_{v,i}^{(u)}}{\partial x_{w,j}}(\mathbf{x}) \right)_{i \in \mathcal{S}, j \in \mathcal{S}},$$

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$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} I & \dots & 0 & \dots & 0 \\ \vdots & \ddots & & & \vdots \\ M_u & \dots & 0 & \dots & M_u \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & \dots & I \end{pmatrix},$$

where the (v, w)-block is

$$\frac{\partial x_v^{(u)}}{\partial \mathbf{x}_w}(\mathbf{x}) = \left(\frac{\partial x_{v,i}^{(u)}}{\partial x_{w,j}}(\mathbf{x}) \right)_{i \in \mathcal{S}, j \in \mathcal{S}},$$

Structure of the Jacobian matrices

- The matrix M_u can be derived from the equality of marginal costs and the flow conservation constraint
- There exist $\boldsymbol{\gamma}, \boldsymbol{\theta} \in \mathbb{R}_+^S$ such that $M_u = [\Gamma B - I]\,\Theta$

$$M_u = \left[\begin{pmatrix} \gamma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_S \end{pmatrix} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} - \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} \right] \begin{pmatrix} \theta_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \theta_S \end{pmatrix}$$

- The vectors $\boldsymbol{\gamma}, \boldsymbol{\theta}$ are such that $\sum_i \gamma_i = 1$, $\theta_i = 0$ if and only if $\gamma_i = 0$ and $\frac{1}{2} \leq \theta_i \leq q < 1$ if $\theta_i \neq 0$

Structure of the Jacobian matrices

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$$M_u = \begin{pmatrix} (\gamma_1 - 1)\theta_1 & \gamma_1\theta_2 & \dots & \gamma_1\theta_S \\ \gamma_2\theta_1 & (\gamma_2 - 1)\theta_2 & \dots & \gamma_2\theta_S \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_S\theta_1 & \gamma_S\theta_2 & \dots & (\gamma_S - 1)\theta_S \end{pmatrix}$$

- The vectors $\boldsymbol{\gamma}, \boldsymbol{\theta}$ are such that $\sum_i \gamma_i = 1$, $\theta_i = 0$ if and only if $\gamma_i = 0$ and $\frac{1}{2} \leq \theta_i \leq q < 1$ if $\theta_i \neq 0$

One round Jacobian

- One round Jacobian: $\mathrm{D}\hat{x}^{(1)}(\mathbf{x}) = \mathrm{D}x^{(K)}(\mathbf{x}) \ldots \mathrm{D}x^{(1)}(\mathbf{x})$
- Example for two players

$$\mathrm{D}\hat{x}^{(1)}(\mathbf{x}) = \begin{pmatrix} I & 0 \\ M_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & M_1 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & M_1 \\ 0 & M_2 M_1 \end{pmatrix}$$

- Example for three players

$$\begin{aligned} J &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ M_3 & M_3 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ M_2 & 0 & M_2 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & M_1 & M_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & M_1 & M_1 \\ 0 & M_2 M_1 & M_2 M_1 + M_2 \\ 0 & M_3(M_1 + M_2 M_1) & M_3(M_1 + M_2 + M_2 M_1) \end{pmatrix} \end{aligned}$$

Conjecture on JSR

Conjecture

The JSR of the set of one-round Jacobians is strictly smaller than one

- Numerical experiments suggest this is true
- If the conjecture is true, then it implies the convergence of the best-response algorithm for routing games.

Two-player routing game

Two-player routing game

Theorem
For the two-player routing game, the sequential best-response dynamics converges for any initial point $\mathbf{x}_0 \in \mathcal{X}$.

Sketch of proof.

- The *n-round Jacobian* is of the form

$$J^{(n)} = \begin{pmatrix} 0 & M_1^{(n)} \\ 0 & M_2^{(n)} M_1^{(n)} \end{pmatrix} \dots \begin{pmatrix} 0 & M_1^{(1)} \\ 0 & M_2^{(1)} M_1^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & M_1^{(n)} M_2^{(n-1)} \dots M_1^{(1)} \\ 0 & M_2^{(n)} M_1^{(n)} \dots M_1^{(1)} \end{pmatrix}$$

- Show that the JSR of the set $\mathcal{M}$ of matrices of the form $[\Gamma B - I]\ \Theta$ is strictly smaller than 1

Two-player routing game

- Any product M of such matrices is such that
  - The sum over any column is zero,
  - If $M\mathbf{x} = \lambda \mathbf{x}$ for some $\lambda \neq 0$, then $\sum_i x_i = 0$,
  - For any diagonal D, $\rho(M) \leq \rho(DB + M) \leq \|DB + M\|_1$.
- With $d_{i,i} = -\min_j (m_{i,j})$ or $d_{i,i} = -\max_j (m_{i,j})$, we obtain

$$\rho(M) \leq \mu_{min}(M) = -\sum_i \min_j (m_{i,j})$$

$$\rho(M) \leq \mu_{max}(M) = \sum_i \max_j (m_{i,j})$$

- Induction: if $M = X[\Gamma B - I]\Theta$

$$\mu_{min}(M) \leq \theta_{max}\, \mu_{max}(X) \quad \text{and} \quad \mu_{max}(M) \leq \theta_{max}\, \mu_{min}(X).$$

Two-player routing game

- If $M = \prod_{k=1}^{n} \left[\Gamma^{(k)} B - I\right] \Theta^{(k)}$, then

$$\rho\left(M\right) \leq \prod_{k=1}^{n} \theta_{max}^{(k)} \leq q^{n}$$

- The JSR of $\mathcal{M}$ is upper bounded by $q < 1$
- Same result proved by Mertzios [Mer09] using a potential function argument.

Conclusion

Conclusion

- Convergence of the sequential BR dynamics for atomic routing games over parallel links
  - ✔ Systematic approach relying only on the structure of the Jacobian matrices
  - ✔ We conjecture that the JSR of the set of one-round Jacobians is strictly smaller than one
  - ✔ Proof of convergence in the case of two players
- Future works
  - ✔ Convergence for the routing game over two links for an arbitrary number of players
  - ✔ Show that the conjecture is true in the general case
  - ✔ Extension to more general networks and to more complex BR dynamics

Questions?

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Two-link routing game

- If $M' = (\Gamma' B - I)\,\Theta'$ and $M = (\Gamma B - I)\,\Theta$, then $M\,M' = tr(M)\,M'$
- For 3 players, the one-round Jacobian is of the form $J = A_1 \otimes M_1 + A_2 \otimes M_2$, where

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & r_2 & r_2 \\ 0 & r_3(1+r_2) & r_3(1+r_2) \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & r_3 \end{pmatrix}$$

- The Jacobian over n rounds if of the form

$$J^{(n)} = Z^{(n)} \dots Z^{(2)} A_1 \otimes M_1 + Z^{(n)} \dots Z^{(2)} A_2 \otimes M_2$$

where $Z^{(n)} = r_1^{(n)} A_1^{(n)} + r_2^{(n)} A_2^{(n)}$